

# List 3-dynamic coloring of graphs with small maximum average degree

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September 21, 2016

## Abstract

An  $r$ -dynamic  $k$ -coloring of a graph  $G$  is a proper  $k$ -coloring  $\phi$  such that for any vertex  $v$ ,  $v$  has at least  $\min\{r, \deg_G(v)\}$  distinct colors in  $N_G(v)$ . The  $r$ -dynamic chromatic number  $\chi_r^d(G)$  of a graph  $G$  is the least  $k$  such that there exists an  $r$ -dynamic  $k$ -coloring of  $G$ . The list  $r$ -dynamic chromatic number of a graph  $G$  is denoted by  $ch_r^d(G)$ .

Recently, Loeb et al. [6] showed that the list 3-dynamic chromatic number of a planar graph is at most 10. And Cheng et al. [2] studied the maximum average condition to have  $\chi_3^d(G) \leq 4, 5$ , or 6. On the other hand, Song et al. [8] showed that if  $G$  is planar with girth at least 6, then  $\chi_r^d(G) \leq r + 5$  for any  $r \geq 3$ .

In this paper, we study list 3-dynamic coloring in terms of maximum average degree. We show that  $ch_3^d(G) \leq 6$  if  $mad(G) < \frac{18}{7}$ , and  $ch_3^d(G) \leq 7$  if  $mad(G) < \frac{14}{5}$ , and both of the bounds are tight. We also show that if  $mad(G) < 3$ , then  $ch_3^d(G) \leq 8$ .

## 1 Introduction

A proper  $k$ -coloring  $\phi : V(G) \rightarrow \{1, 2, \dots, k\}$  of a graph  $G$  is an assignment of colors to the vertices of  $G$  so that any two adjacent vertices receive distinct colors. The chromatic number  $\chi(G)$  of a graph  $G$  is the least  $k$  such that there exists a proper  $k$ -coloring of  $G$ . An  $r$ -dynamic  $k$ -coloring of a graph  $G$  is a proper  $k$ -coloring  $\phi$  such that for each vertex  $v \in V(G)$ , either the number of distinct colors in its neighborhood is at least  $r$  or the colors in its neighborhood are all distinct, that is,  $|\phi(N_G(v))| = \min\{r, \deg_G(v)\}$ . The  $r$ -dynamic chromatic number  $\chi_r^d(G)$  of a graph  $G$  is the least  $k$  such that there exists an  $r$ -dynamic  $k$ -coloring of  $G$ .

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\*This research was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education(NRF-2015R1D1A1A01057008) (S.-J. Kim)

<sup>†</sup>This research was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Science, ICT and Future Planning(2015R1C1A1A01053495) (B. Park)

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A *list assignment* on a graph  $G$  is a function  $L$  that assigns each vertex  $v$  a set  $L(v)$  which is a list of available colors at  $v$ . For a list assignment  $L$  of a graph  $G$ , we say  $G$  is  *$L$ -colorable* if there exists a proper coloring  $\phi$  such that  $\phi(v) \in L(v)$  for every  $v \in V(G)$ . A graph  $G$  is said to be  *$k$ -choosable* if for any list assignment  $L$  such that  $|L(v)| \geq k$  for every vertex  $v$ ,  $G$  is  $L$ -colorable. The *list chromatic number*  $\chi_\ell(G)$  of a graph  $G$  is the least  $k$  such that  $G$  is  $k$ -choosable.

For a list assignment  $L$  of  $G$ , we say  $G$  is  *$r$ -dynamic  $L$ -colorable* if there exists an  $r$ -dynamic coloring  $\phi$  such that  $\phi(v) \in L(v)$  for every  $v \in V(G)$ . A graph  $G$  is  *$r$ -dynamic  $k$ -choosable* if for any list assignment  $L$  with  $|L(v)| \geq k$  for every vertex  $v$ ,  $G$  is  $r$ -dynamic  $L$ -colorable. The *list  $r$ -dynamic chromatic number*  $ch_r^d(G)$  of a graph  $G$  is the least  $k$  such that  $G$  is  $r$ -dynamic  $k$ -choosable.

Note that  $r$ -dynamic coloring was studied in [4, 5], and was also studied in [2, 7, 8] with the name of  *$r$ -hued coloring*. Similar to the Wegner's conjecture [9], a conjecture about dynamic coloring of planar graphs was proposed in [7].

**Conjecture 1.1.** *Let  $G$  be a planar graph. Then*

$$\chi_r^d(G) \leq \begin{cases} r+3 & \text{if } 1 \leq r \leq 2 \\ r+5 & \text{if } 3 \leq r \leq 7 \\ \lfloor \frac{3r}{2} \rfloor + 1 & \text{if } r \geq 8. \end{cases}$$

Song, Lai, and Wu [8] showed that Conjecture 1.1 is true for planar graphs with girth at least 6.

**Theorem 1.2** ([8]). *If  $G$  is a planar graph with girth at least 6,  $\chi_r^d(G) \leq r+5$  for any  $r \geq 3$ .*

Recently, 3-dynamic coloring has been concerned. Loeb, Mahoney, Reiniger, and Wise [6] showed that  $ch_3^d(G) \leq 10$  if  $G$  is a planar graph. On the other hand, list 3-dynamic coloring was studied in [2] in terms of maximum average degree, where the *maximum average degree* of a graph  $G$ ,  $mad(G)$ , is the maximum among the average degrees of the subgraphs of  $G$ . It was showed in [2] that  $\chi_3^d(G) \leq 6$  if  $mad(G) < \frac{12}{5}$ ,  $\chi_3^d(G) \leq 5$  if  $mad(G) < \frac{7}{3}$ , and  $\chi_3^d(G) \leq 4$  if  $G$  has no  $C_5$ -component and  $mad(G) < \frac{8}{3}$ .

In this paper, we study list 3-dynamic coloring with maximum average degree condition. For each  $k \in \{6, 7, 8\}$ , we study the optimal value of maximum average degree to be  $ch_3^d(G) \leq k$ . First, we give an optimal value of  $mad(G)$  to be  $ch_3^d(G) \leq 6$ , which improves a result in [2].

**Theorem 1.3.** *If  $mad(G) < \frac{18}{7}$ , then  $ch_3^d(G) \leq 6$ .*

The bound on  $mad(G)$  in Theorem 1.3 is tight. The graph  $G$  in Figure 1 is a subcubic graph and so  $ch_3^d(G) = ch(G^2)$ , where the *square* of  $G$ ,  $G^2$ , is the graph obtained by adding to  $G$  the edges connecting two vertices having a common neighbor in  $G$ . Note that  $mad(G) = \frac{18}{7}$  and  $G^2$  is isomorphic to  $K_7$ . Hence we have  $ch(G^2) = ch_3^d(G) = 7$ , which implies that the bound on  $mad(G)$  in Theorem 1.3 is tight.

We also study the value of  $mad(G)$  to be  $ch_3^d(G) \leq 7$ .

**Theorem 1.4.** *If  $mad(G) < \frac{14}{5}$ , then  $ch_3^d(G) \leq 7$ .*

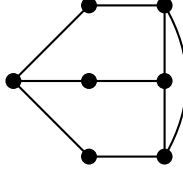


Figure 1: A tight example for Theorem 1.3,  $mad(G) = \frac{18}{7}$  and  $\chi_3^d(G) = 7$

Let  $G$  be the graph that is obtained from the Petersen graph by deleting one edge. Then  $mad(G) < \frac{14}{5}$  and  $ch_3^d(G) = 8$ . Thus the bound in Theorem 1.4 is tight.

We also show that any graph  $G$  is 3-dynamic 8-choosable if  $mad(G) < 3$ .

**Theorem 1.5.** *If  $mad(G) < 3$ , then  $ch_3^d(G) \leq 8$ .*

Note that every planar graph  $G$  with girth at least  $g$  satisfies  $mad(G) < \frac{2g}{g-2}$ . Thus from Theorem 1.3, Theorem 1.4, and Theorem 1.5, we have the following corollary. Note that Theorem 1.5 implies Theorem 1.2 when  $r = 3$

**Corollary 1.6.** *Let  $G$  be a planar graph. Then we have that*

- (1)  $ch_3^d(G) \leq 6$  if the girth of  $G$  is at least 9,
- (2)  $ch_3^d(G) \leq 7$  if the girth of  $G$  is at least 7,
- (3)  $ch_3^d(G) \leq 8$  if the girth of  $G$  is at least 6.

It was showed in [3] that  $ch(G) \leq 6$  if  $mad(G) < \frac{18}{7}$  and  $\Delta(G) \leq 3$ . And it was also showed in [1] independently that  $ch(G) \leq 6$  if  $mad(G) < \frac{18}{7}$ ,  $\Delta(G) \leq 3$ , and the girth of  $G$  is at least 7. Thus Theorem 1.3 is an extension of the results in [1, 3]. On the other hand, it was showed in [1] that  $ch(G) \leq 7$  if  $mad(G) < \frac{14}{5}$  and  $\Delta(G) \leq 3$ . Thus Theorem 1.4 is an extension of the result in [1]. Consequently, Corollary 1.6 is an extension of the results in [1].

This paper is organized as follows. In Section 2, we give preliminaries about simple reducible configurations. In Sections 3, 4, and 5, we prove Theorems 1.3, 1.4, and 1.5, respectively.

## 2 Preliminaries

A vertex of degree  $d$  is called a  $d$ -vertex, and a vertex of degree at least  $d$  (at most  $d$ ) is called a  $d^+$ -vertex ( $d^-$ -vertex). If  $x$  is adjacent to a  $d$ -vertex  $y$  ( $d^+$ -vertex, or  $d^-$ -vertex), then we say that  $y$  is a  $d$ -neighbor of  $x$  ( $d^+$ -neighbor of  $x$ , or  $d^-$ -neighbor of  $x$ ). Two vertices  $x$  and  $y$  are *weakly adjacent* in  $G$  if there is a 2-vertex  $z$  such that  $xz, zy$  are edges of  $G$ , and in this case, we say  $x$  is a *weak neighbor* of  $y$ .

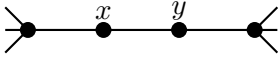
For each  $i \in \{0, 1, 2, 3\}$ , we let  $W_i(G)$  be the set of 3-vertices which have exactly  $i$  2-neighbors. That is,

$$W_i(G) = \{v \in V(G) \mid \deg(v) = 3, \text{ and exactly } i \text{ neighbors of } v \text{ are 2-vertices}\}.$$

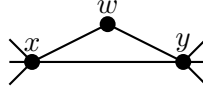
If there is no confusion, we denote  $W_i(G)$  by  $W_i$ . And let  $[n] = \{1, 2, \dots, n\}$ .

**Lemma 2.1.** *Let  $k \geq 6$ . Let  $G$  a graph with the smallest number of vertices and edges such that  $ch_3^d(G) > k$ . Then the followings hold.*

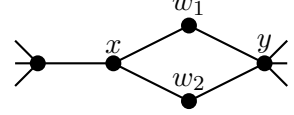
- (1) *There is no  $1^-$ -vertex.*
- (2) *No two  $2^-$ -vertices are not adjacent.*
- (3) *For any edge  $xy \in E(G)$ , there is no 2-vertex which is adjacent to both of  $x$  and  $y$ .*
- (4) *For each  $i \in [3]$ , every vertex in  $W_i(G)$  has  $i$  distinct weak neighbors.*



(a) Figure for (2)



(b) Figure for (3)



(c) Figure for (4)

Figure 2: Figures for Lemmas 2.1

*Proof.* We prove (1)~(4) one by one. Since  $ch_3^d(G) > k$ , there is a list assignment  $L$  of  $G$  such that  $|L(v)| \geq k$  for each vertex  $v$  of  $G$ , and  $G$  is not 3-dynamic  $L$ -colorable.

(1) Let  $v$  be a  $1^-$ -vertex. Since  $H = G - \{v\}$  is smaller than  $G$ ,  $H$  is 3-dynamic  $L$ -colorable. Thus there is a 3-dynamic coloring  $\phi$  of  $H$  such that  $\phi(a) \in L(a)$  for any  $a \in V(H)$ . Note that the number of available colors at  $v$  is at least  $k - 3$ . Since  $k - 3 \geq 1$ , it is easy to see that  $\phi$  is extended to a 3-dynamic coloring of  $G$  so that  $G$  is 3-dynamic  $L$ -colorable, which is a contradiction to the choice of  $G$ .

(2) Suppose that two 2-vertices  $x$  and  $y$  are adjacent (See Figure 2a). Let  $H = G - \{x, y\}$ . Then  $H$  is 3-dynamic  $L$ -colorable since  $H$  is smaller than  $G$ . Thus there is a 3-dynamic coloring  $\phi$  of  $H$  such that  $\phi(a) \in L(a)$  for any  $a \in V(H)$ . Note that the number of available colors at  $x$  and  $y$  are at least  $k - 4$ . And since  $k - 4 \geq 2$ , it is easy to see that  $\phi$  is extended to a 3-dynamic coloring of  $G$  so that  $G$  is 3-dynamic  $L$ -colorable, a contradiction to the choice of  $G$ .

(3) Suppose that for an edge  $xy \in E(G)$ , there is a common neighbor  $w$  of  $x$  and  $y$ , which is a 2-vertex (See Figure 2b). Let  $H = G - \{w\}$ . Then  $H$  is 3-dynamic  $L$ -colorable since  $H$  is smaller than  $G$ . Thus there is a 3-dynamic coloring  $\phi$  of  $H$  such that  $\phi(a) \in L(a)$  for any  $a \in V(H)$ . Note that the number of available colors at  $w$  is at least  $k - 4$ . Since  $k - 4 \geq 1$ , it is easy to see that  $\phi$  is extended to a 3-dynamic coloring of  $G$  so that  $G$  is 3-dynamic  $L$ -colorable, a contradiction to the choice of  $G$ .

(4) From (3), we know that for any vertex, the set of neighbors and the set of weak neighbors are disjoint. It is clear that the lemma is true then  $i \geq 1$ . Suppose that there is a vertex  $x \in W_2 \cup W_3$  such that  $x$  has two 2-neighbors  $w_1$  and  $w_2$  and the other neighbors of  $w_1$  and  $w_2$  are the same as a vertex  $y$  (See Figure 2c). Let  $H = G - \{w_1\}$ .  $H$  is 3-dynamic  $L$ -colorable, since  $H$  is smaller than  $G$ . Thus there is a 3-dynamic coloring  $\phi$  of  $H$  such that  $\phi(a) \in L(a)$  for any  $a \in V(H)$ . Note that the number of available colors at  $w_1$  is at least  $k - 5$ . Since  $k - 5 \geq 1$ , it is easy

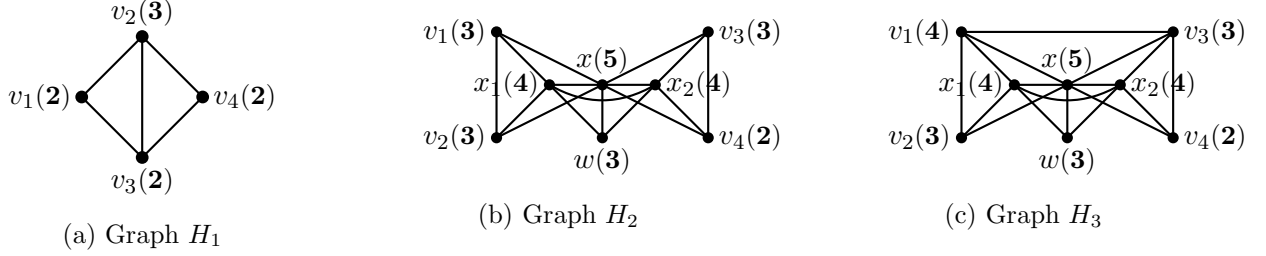


Figure 3: Graphs in Remark 2.2. The bold number in the parenthesis of each vertex in graph  $H_i$  denotes the value of  $f_i$  in Remark 2.2.

to see that  $\phi$  is extended to a 3-dynamic coloring of  $G$  so that  $G$  is 3-dynamic  $L$ -colorable, a contradiction to the choice of  $G$ .  $\square$

The following are simple properties in list coloring, which will be often used in the paper. For a function  $f$  assigning a positive integer to each  $v \in V(G)$ , a graph  $G$  is said to be  $f$ -choosable if for any list assignment  $L$  such that  $|L(v)| \geq f(v)$  for every vertex  $v$ ,  $G$  is  $L$ -colorable.

**Remark 2.2.** For each  $i \in [3]$ , graph  $H_i$  in Figure 3 are  $f_i$ -choosable.

- (a) Let  $H_1 = K_4 - v_1v_4$  with  $V(H_1) = \{v_1, v_2, v_3, v_4\}$ , which is the graph in (a) of Figure 3. Let  $f_1(v_1) = 2$ ,  $f_1(v_2) = 3$ ,  $f_1(v_3) = 2$ , and  $f_1(v_4) = 2$ .

*Proof.* If  $L(v_1) \cap L(v_4) \neq \emptyset$ , then color  $v_1$  and  $v_4$  with a color  $c \in L(v_1) \cap L(v_4)$ . And then color  $v_3$  and  $v_2$ . If  $L(v_1) \cap L(v_4) = \emptyset$ , then color  $v_2$  with a color  $c \notin L(v_3)$ . And then, the number of available colors at the remaining three vertices in the path are 1, 2, 2. In each case, we can see that  $H_1$  is  $f_1$ -choosable.  $\square$

- (b) Let  $H_2$  be a graph with  $V(H_2) = \{v_1, v_2, v_3, v_4, x_1, x_2, w\}$ , which is the graph in (b) of Figure 3. Let  $f_2(v_1) = f_2(v_2) = f_2(v_3) = 3$ ,  $f_2(v_4) = 2$ ,  $f_2(x_j) = 4$ ,  $f_2(x) = 5$ ,  $f_2(w) = 3$ .

*Proof.* First color the vertex  $x$  with a color  $c \notin L(v_1)$ . And then color the remained vertices in the order of  $v_4, v_3, x_2, w, x_1, v_2, v_1$ .  $\square$

- (c) Let  $H_3$  be a graph with  $V(H_3) = \{v_1, v_2, v_3, v_4, x_1, x_2, w\}$ , which is the graph in (c) of Figure 3. Let  $f_3(v_1) = f_3(v_2) = f_3(v_3) = 3$ ,  $f_3(v_4) = 2$ ,  $f_3(x_j) = 4$ ,  $f_3(x) = 5$ ,  $f_3(w) = 3$ .

*Proof.* First color the vertex  $x$  with a color  $c \notin L(v_1)$ . And then color the remained vertices in the order of  $v_4, v_3, x_2, w, x_1, v_2, v_1$ .  $\square$

**Lemma 2.3.** Let  $k \geq 6$ . Let  $G$  be a graph with the smallest number of vertices and edges such that  $ch_3^d(G) > k$ . The graphs in Figure 4 do not appear as an induced subgraph in  $G$ .

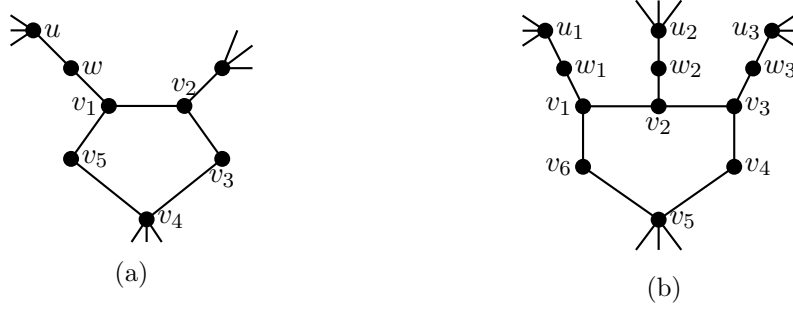


Figure 4: Graphs in Lemma 2.3 (All labelled vertices are distinct.)

*Proof.* Let  $L$  be a list assignment of  $G$  such that  $|L(v)| \geq k$  for each vertex  $v$  of  $G$  and  $G$  is not 3-dynamic  $L$ -colorable. Note that all labelled vertices in the figure are distinct. Suppose that the graph in (a) of Figure 4 appears in  $G$  as an induced subgraph. That has distinct 7 vertices and  $v_1, v_2$  are 3-vertices and  $v_3, v_5, w$  are 2-vertices, and  $v_4, u$  are  $3^+$ -vertices. Let  $S = \{v_1, v_3, v_5, w\}$  and  $H = G - S$ . Since  $H$  is smaller than  $G$ ,  $H$  is 3-dynamic  $L$ -colorable. Thus there is a 3-dynamic coloring  $\phi$  of  $H$  such that  $\phi(a) \in L(a)$  for any  $a \in V(H)$ , and  $v_2$  and  $v_4$  get distinct colors in  $\phi$  (we can recolor  $v_2$ ). For each  $a \in S$ , let  $L'(a)$  be a set of available colors at  $a$  such that  $\phi$  is extended to a 3-dynamic coloring of  $G$  so that  $G$  is 3-dynamic  $L$ -colorable. More precisely,  $L'$  is determined excluding forbidden colors of each vertex as follows. (We add explanation only for this case. Throughout all proofs, we use a similar technique for obtaining such  $L'$ , we omit explanation at the other places.) Let  $c_1$  be a color which is colored at the neighbors of  $v_4$  in  $H$ , that is  $c_1 \in \{\phi(x) : x \in N_H(v_4)\}$ . And let  $c_2$  be the color which is assigned at the neighbor of  $v_2$  in  $H$ , that is  $c_2 = \phi(v'_2)$  where  $N_G(v_2) = \{v_1, v_3, v'_2\}$ . We have the following.

- the forbidden colors at  $v_1$  are  $\phi(v_2), \phi(v_4), c_2$ ;
- the forbidden colors at  $v_3$  are  $\phi(v_2), \phi(v_4), c_1, c_2$ ;
- the forbidden colors at  $v_5$  are  $\phi(v_2), \phi(v_4), c_1$ ;
- the forbidden colors at  $w$  are  $\phi(v_2), \phi(u)$ , and two colors from neighbors of  $u$  in  $H$ .

Therefore,

$$|L'(v_1)| \geq k - 4, \quad |L'(v_3)| \geq k - 4, \quad |L'(v_5)| \geq k - 3, \quad |L'(w)| \geq k - 4.$$

Note that the subgraph of  $G^2$  induced by  $S$ ,  $G^2[S]$ , is isomorphic to  $K_4$  minus an edge  $wv_3$ , a graph in (a) of Figure 3. Since  $k - 4 \geq 2$  and  $k - 3 \geq 3$ ,  $G^2[S]$  is  $L'$ -colorable by (a) of Remark 2.2. Then it is easy to see that  $\phi$  is extended to a 3-dynamic coloring of  $G$  so that  $G$  is 3-dynamic  $L$ -colorable, a contradiction.

Next, suppose that  $G$  has the graph in (b) of Figure 4 as an induced subgraph. That has distinct 12 vertices and  $v_1, v_2, v_3$  are 3-vertices and  $v_4, v_6, w_1, w_2, w_3$  are 2-vertices, and  $v_5, u_1, u_2, u_3$  are  $3^+$ -vertices. Let  $S = \{v_1, v_2, v_3, v_4, v_6, w_1, w_2, w_3\}$ . Let  $H = G - S$ . Since  $H$  is smaller than  $G$ ,  $H$  is 3-dynamic  $L$ -colorable. Thus there is a 3-dynamic coloring  $\phi$  of  $H$  such that  $\phi(a) \in L(a)$  for any  $a \in V(H)$ . For each  $a \in S$ , let  $L'(a)$  be a set of available colors at  $a$

such that  $\phi$  is extended to a 3-dynamic coloring of  $G$  so that  $G$  is 3-dynamic  $L$ -colorable. Note that  $G^2[S]$  is isomorphic to the graph in Figure 3 and

$$|L'(v_2)| \geq k - 1, \quad |L'(v_i)| \geq k - 2 \text{ for } i \in \{1, 3, 4, 6\}, \quad |L'(w_i)| \geq k - 3 \text{ for } i \in \{1, 2, 3\}.$$

Note that we forbid just two colors at  $v_4$  and  $v_6$  since we will color  $v_4$  and  $v_6$  differently. By (c) of Remark 2.2, it is 3-dynamic  $L'$ -colorable. Thus  $G$  is 3-dynamic  $L$ -colorable, a contradiction.  $\square$

### 3 Proof of Theorem 1.3

In this section, we prove Theorem 1.3. We use the induction on the number of vertices and the number of edges. In the following, we let  $G$  be a minimal counterexample to Theorem 1.3. That is,  $G$  is a graph with the smallest number of vertices and edges,  $mad(G) < 18/7$ , and  $ch_3^d(G) \geq 7$ . Then there exists a list assignment  $L$  such that  $|L(v)| \geq 6$  for each  $v \in V(G)$  and  $G$  is not 3-dynamic  $L$ -colorable.

From now on, we show that several subgraphs can not appear in  $G$ , which are called reducible configurations. More precisely, we will show the following [C1]~[C6]:

[C1] There is no  $1^-$  vertex. ((1) of Lemma 2.1)

[C2] No two 2-vertices are adjacent. ((2) of Lemma 2.1)

[C3] No two vertices in  $W_2$  are adjacent. (Lemma 3.1, Figure 5)

[C4] For a vertex  $x \in W_3$ ,  $x$  has three distinct weak neighbors in  $W_1$ . (Lemma 3.2, Figure 6)

[C5] There is no vertex in  $W_1$ , which has two neighbors in  $W_2$ . (Lemma 3.3, Figure 7)

[C6] There is no vertex 3-vertex, which has one neighbor in  $W_1$ , one neighbor in  $W_2$ , one weak neighbor in  $W_3$ . (Lemma 3.4, Figure 8)

Note that [C1] and [C2] hold by Lemma 2.1. We will see that [C3]~[C6] hold.

**Lemma 3.1.** [C3] No two vertices in  $W_2$  are adjacent.

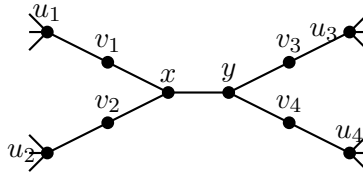


Figure 5: An illustration of [C3] (Lemma 3.1),  $x, y \in W_2$

*Proof.* Suppose that there are two vertices  $x$  and  $y$  in  $W_2$  that are adjacent. That is,  $x$  and  $y$  are 3-vertices and both of  $x$  and  $y$  have exactly two 2-neighbors. Let  $v_1, v_2$  be the 2-neighbors of  $x$  and let  $v_3, v_4$  be the 2-neighbors of  $y$ . Let  $u_i$  be the  $3^+$ -neighbor of  $v_i$  for  $i \in \{1, 2, 3, 4\}$  (see Figure 5).

By (4) of Lemma 2.1,  $u_1 \neq u_2$  and  $u_3 \neq u_4$ . If  $u_1 = u_3$ , then  $x, y, v_3, u_1, v_1, v_2$  form the induced subgraph (a) in Figure 4, this is impossible by Lemma 2.3. Thus  $u_1, u_2, u_3, u_4$  are all distinct. Let  $S = \{x, y, v_1, v_2, v_3, v_4\}$  and let  $H = G - S$ . Since  $G$  is a minimal counterexample and  $H$  is smaller than  $G$ ,  $H$  is 3-dynamic  $L$ -colorable. Thus there is a 3-dynamic coloring  $\phi$  of  $H$  such that  $\phi(a) \in L(a)$  for any  $a \in V(H)$ . For each  $a \in S$ , let  $L'(a)$  be a set of available colors at  $a$  such that  $\phi$  is extended to a 3-dynamic coloring of  $G$  so that  $G$  is 3-dynamic  $L$ -colorable. Then

$$|L'(x)| \geq 4, \quad |L'(y)| \geq 4, \quad \text{and} \quad |L'(v_i)| \geq 3 \text{ for } i \in [4].$$

Since  $|L'(v_1)| + |L'(v_3)| > |L'(x)|$ , we can give colors to  $v_1$  and  $v_3$  so that the number of available colors remained at  $x$  is at least 3. Then the remained vertices  $x, y, v_2, v_4$  form  $K_4$  minus an edge  $v_2v_4$  as in Figure 3, and the numbers of available colors are 3, 2, 2, 2, respectively. By (a) of Remark 2.2, it is colorable. This implies that  $G$  is 3-dynamic  $L$ -colorable, a contradiction.  $\square$

From (4) of Lemma 2.1, a vertex in  $W_3$  has three weak neighbors.

**Lemma 3.2.** [C4] If  $x \in W_3$  and  $y$  is a weak neighbor of  $x$ , then  $y \in W_1$ . That is,  $x$  has three weak neighbors in  $W_1$ .

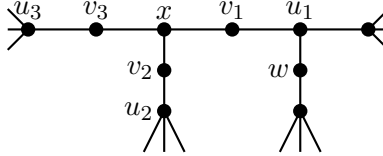


Figure 6: An illustration of [C4] (Lemma 3.2),  $x \in W_3$ ,  $u_1 \notin W_1$

*Proof.* Let  $x$  be a 3-vertex in  $W_3$  and let  $v_1, v_2, v_3$  be the 2-neighbors of  $x$ . Let  $u_i$  be the other neighbor of  $v_i$  for each  $i \in [3]$  (so they are weak neighbors of  $x$ ). By [C2], each  $u_i$  is a  $3^+$ -vertex. By (4) of Lemma 2.1, three vertices  $u_1, u_2, u_3$  are distinct.

First, we will show that  $u_i$  is a 3-vertex for each  $i \in [3]$ . Suppose that some  $u_i$  is not a 3-vertex for some  $i \in [3]$ . Without loss of generality, assume that  $u_1$  is not a 3-vertex. Then  $u_1$  is a  $4^+$ -vertex by [C2]. Let  $S = \{x, v_1, v_2, v_3\}$  and  $H = G - S$ . Then  $H$  is 3-dynamic  $L$ -colorable, since  $G$  is a minimal counterexample and  $H$  is smaller than  $G$ . Thus there is a 3-dynamic coloring  $\phi$  of  $H$  such that  $\phi(a) \in L(a)$  for any  $a \in V(H)$ . For each  $a \in S$ , let  $L'(a)$  be a set of available colors at  $a$  such that  $\phi$  is extended to a 3-dynamic coloring of  $G$  so that  $G$  is 3-dynamic  $L$ -colorable. Then

$$|L'(v_1)| \geq 5, \quad |L'(v_2)| \geq 3, \quad |L'(v_3)| \geq 3 \quad \text{and} \quad |L'(x)| \geq 3.$$

Since  $G^2[S]$  is  $K_4$ ,  $G^2[S]$  is  $f_S$ -choosable. This implies that  $\phi$  is extended to a 3-dynamic coloring of  $G$  so that  $G$  is 3-dynamic  $L$ -colorable, a contradiction. Hence  $u_i$  is a 3-vertex for each  $i \in [3]$ .



Next, we will show that every  $u_i$  is in  $W_1$ , which means that  $x$  is the only weak neighbor of  $u_1$ . Without loss of generality, we may assume that  $u_1$  has another 2-neighbor  $w$  other than  $v_1$ . Then all 8 vertices  $x, v_i$ 's,  $u_i$ 's,  $w$  are distinct (See Figure 6). Note that  $x$  and  $w$  cannot have a common neighbor as all neighbors of  $x$  are  $v_i$ 's. Let  $S = \{x, v_1, u_1, w\}$ . Since  $G$  is a minimal counterexample and  $H$  is smaller than  $G$ ,  $H$  is 3-dynamic  $L$ -colorable. Thus there is a 3-dynamic coloring  $\phi$  of  $H$  such that  $\phi(a) \in L(a)$  for any  $a \in V(H)$ . For each  $a \in S$ , let  $L'(a)$  be a set of available colors at  $a$  such that  $\phi$  is extended to a 3-dynamic coloring of  $G$  so that  $G$  is 3-dynamic  $L$ -colorable. Then  $G^2[S]$  is the graph in (a) of Figure 3 and

$$|L'(x)| \geq 2, |L'(v_1)| \geq 3, |L'(u_1)| \geq 2, |L'(w)| \geq 2.$$

By (a) of Remark 2.2,  $G^2[S]$  is  $L'$ -colorable. This implies that  $G$  is 3-dynamic  $L$ -colorable, a contradiction.  $\square$

**Lemma 3.3.** [C5] There is no vertex in  $W_1$  which has two neighbors in  $W_2$ .

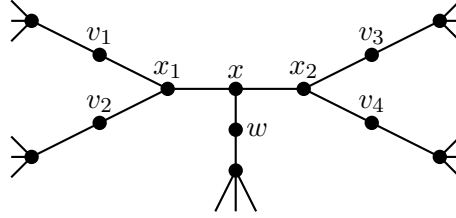


Figure 7: An illustration of [C5] (Lemma 3.3),  $x \in W_1$ ,  $x_1, x_2 \in W_2$

*Proof.* Suppose that there is a vertex  $x \in W_1$  such that  $x$  has two neighbors  $x_1, x_2 \in W_2$ . Then all 5 vertices in  $N_G(x) \cup N_G(x_1) \cup N_G(x_2) - \{x_1, x_2\}$  are 2-vertices. We label those vertices as in Figure 7. Let  $S = \{x, x_1, x_2, v_1, v_2, v_3, v_4, w\}$ . Let  $u_i$  be the neighbor of  $v_i$  other than  $v_i$  for each  $i \in [4]$ , and  $w'$  be the neighbor of  $w$  other than  $x$ . Note that  $u_1 \neq u_2$  and  $u_3 \neq u_4$  by (4) of Lemma 2.1.

Then  $w' \neq u_1$ , otherwise the five vertices  $w', w, v_1, x_1, x$  form a cycle  $C_5$  and together with the three vertices  $v_2, u_2, x_2$ , they form the induced subgraph (a) in Figure 4. Hence,  $w' \neq u_i$  for all  $i \in [4]$ . If  $u_1 = u_3$ , then we have the graph (b) in Figure 4, which is a contradiction. Thus  $u_1, u_2, u_3, u_4$ , and  $w'$  are distinct.

Let  $H = G - S$ . Since  $G$  is a minimal counterexample and  $H$  is smaller than  $G$ ,  $H$  is 3-dynamic  $L$ -colorable. Thus there is a 3-dynamic coloring  $\phi$  of  $H$  such that  $\phi(a) \in L(a)$  for any  $a \in V(H)$ . For each  $a \in S$ , let  $L'(a)$  be a set of available colors at  $a$  such that  $\phi$  is extended to a 3-dynamic coloring of  $G$  so that  $G$  is 3-dynamic  $L$ -colorable. Then

$$|L'(x)| \geq 5, |L'(x_1)| \geq 4, |L'(x_2)| \geq 4, |L'(w)| \geq 3, |L'(v_i)| \geq 3 \text{ (for } i \in [4]),$$

and  $G^2[S]$  is graph  $H_2$  in Figure 3. By (b) of Remark 2.2, it is colorable. This implies that  $G$  is 3-dynamic  $L$ -colorable, a contradiction.  $\square$

**Lemma 3.4.** [C6] There is no 3-vertex which has one neighbor in  $W_1$ , one neighbor in  $W_2$ , and one weak neighbor in  $W_3$ .

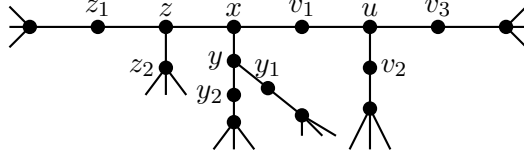


Figure 8: An illustration of [C6] (Lemma 3.4),  $x, z \in W_1$ ,  $y \in W_2$ ,  $u \in W_3$

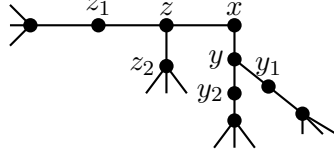


Figure 9: The local structure of  $H = G - \{v_1, v_2, u\}$  near the vertex  $x$

*Proof.* Suppose that there exists a 3-vertex  $x$  which has one neighbor  $y$  in  $W_2$ , one neighbor  $z$  in  $W_1$ , one weak neighbor  $u$  in  $W_3$ . Let  $v_1$  be the 2-neighbor of  $x$ , let  $v_2$  and  $v_3$  be the other 2-neighbors of  $u$ . Let  $y_1$  and  $y_2$  be two 2-neighbors of  $y$  other than  $v_1$ , and let  $z_1$  and  $z_2$  be two neighbors of  $z$  other than  $x$  (See Figure 8).

Since  $H = G - \{v_1, v_2, u\}$  is smaller than  $G$ , there is a 3-dynamic coloring  $\phi$  of  $H$  such that  $\phi(a) \in L(a)$  for any  $a \in V(H)$ . In the graph  $H$ , the vertex  $x$  can be recolored without changing the colors of the other vertices, except 5 vertices  $y, z, y_1, y_2, z_1$  (see Figure 9) by Claim 3.5. (The following claim appeared in Lemma 17 in [1]. But, we include here for the sake of completeness.)

**Claim 3.5.** *There is a 3-dynamic coloring  $\phi'$  of  $H$  such that  $\phi'(a) \in L(a)$  for all  $a \in V(H)$ , and  $\phi(x) \neq \phi'(x)$ ,  $\phi(a) = \phi'(a)$  for any vertex  $a \in V(G) \setminus \{y, z, y_1, y_2, z_1\}$ .*

*Proof of Claim 3.5.* We uncolor the colors of 6 vertices  $x, y, z, y_1, y_2, z_1$  from  $\phi$ . Then we will show that we can recolor the vertices so that the new color of  $x$  is not different from  $\phi(x)$ . For each vertex  $v$  in  $S$ , let  $L'(v)$  be the set of available colors at  $v$  to make  $\phi$  extended to a 3-dynamic coloring of  $G$  so that  $G$  is 3-dynamic  $L$ -colorable. Then

$$|L'(x)| \geq 5, |L'(y)| \geq 4, |L'(z)| \geq 2, |L'(y_1)| \geq 3, |L'(y_2)| \geq 3, |L'(z_1)| \geq 2.$$

Color  $y$  by a color  $c \notin L'(y_1)$ . Redefine  $L'(v)$  by the set of available colors at  $v$  after coloring  $y$ . Then

$$|L'(x)| \geq 4, |L'(z)| \geq 1, |L'(y_1)| \geq 3, |L'(y_2)| \geq 2, |L'(z_1)| \geq 2.$$

Color  $z$  and  $z_1$ , then redefine  $L'(v)$  by the set of available colors at  $v$  after coloring  $z$  and  $z_1$

$$|L'(x)| \geq 2, |L'(y_1)| \geq 3, |L'(y_2)| \geq 2,$$

the number of available colors remained at  $x$  is at least 2. Thus we can recolor  $x$  with a color distinct from  $\psi(x)$ . This completes the proof of Claim 3.5.  $\square$

For each  $a \in S$ , let  $L'(a)$  be a set of available colors at  $a$  such that  $\phi$  is extended to a 3-dynamic coloring of  $G$  so that  $G$  is 3-dynamic  $L$ -colorable. Then

$$|L'(v_1)| \geq 2, \quad |L'(v_2)| \geq 2, \quad |L'(u)| \geq 2.$$

Let  $u_2$  and  $u_3$  be the neighbors of  $v_2$  and  $v_3$  other than  $u$ , respectively. Select and fix two colors  $c_1$  and  $c_2$  in  $\{\phi(q) : q \in N_G(u_2) \setminus \{v_2\}\}$ . Then we have the following.

- the forbidden colors at  $v_1$  are  $\phi(v_3), \phi(x), \phi(y), \phi(z)$ ;
- the forbidden colors at  $v_2$  are  $\phi(v_3), \phi(u_2), c_1, c_2$ ;
- the forbidden colors at  $u$  are  $\phi(v_3), \phi(x), \phi(u_2), \phi(u_3)$ .

By Claim 3.5, we can assume that a set of available colors at  $v_2$  is not equal to that of  $u$  by recoloring  $x$ . As each has two available colors and all of them are not same, we can color  $v_1, v_2, u$  from the lists. Thus  $G$  is 3-dynamic  $L$ -colorable, a contradiction.  $\square$

We use discharging technique. We define the charge of each vertex  $v$  of  $G$  by its degree  $\deg(v)$ . Note that the average charge is less than  $\frac{18}{7}$ . Next, we distribute their charges by the following rules, and then we show that the new charge of each vertex is at least  $\frac{18}{7}$ , which leads a contradiction.

Recall that  $W_2$  is the set of 3-vertices which have two 2-neighbors, and  $W_3$  is the set of 3-vertices which have three 2-neighbors. See also Figure 10 for discharging rules.

### Discharging Rules

- R1.** A  $3^+$ -vertex gives  $\frac{2}{7}$  to each of its 2-neighbors.
- R2.** A  $3^+$ -vertex gives  $\frac{1}{7}$  to each of its weak neighbors in  $W_3$ .
- R3.** A  $3^+$ -vertex gives  $\frac{1}{7}$  to each of its neighbors in  $W_2$ .
- R4.** A 3-vertex in  $W_0$  gives  $\frac{1}{7}$  to each of its neighbors  $x$  in  $W_1$  if  $x$  has a neighbor in  $W_2$  and a weak neighbor in  $W_3$ .

Let  $d^*(u)$  be the new charge after discharging. We will show that  $d^*(u) \geq \frac{18}{7}$  for all  $u \in V(G)$ . Note that by [C1] each vertex of  $G$  is a  $2^+$ -vertex.

(1) Suppose that  $\deg(u) = 2$ . By [C2] the neighbors of  $u$  are  $3^+$ -vertices and so it receives  $\frac{2}{7}$  from each of its neighbors by **R1**, and so the new charge of  $u$  is

$$d^*(u) = 2 + \frac{2}{7} + \frac{2}{7} = \frac{18}{7}.$$

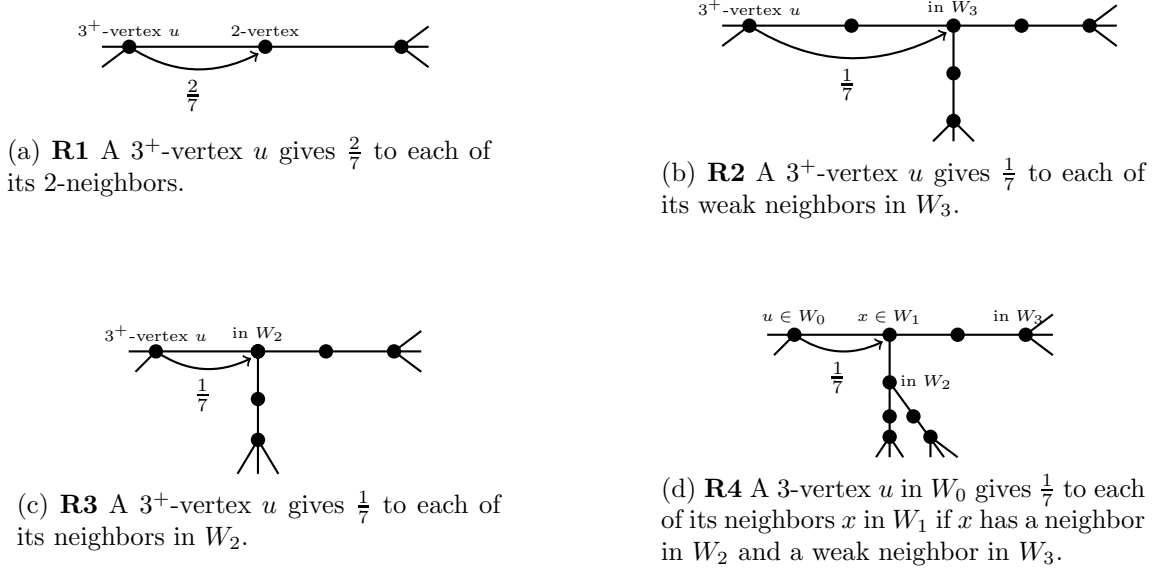


Figure 10: An illustration of Discharging Rules

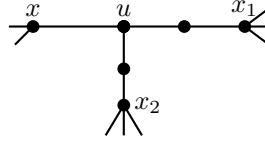


Figure 11: An illustration of case (2-2),  $u \in W_2$

(2) Suppose that  $\deg(u) = 3$ . If  $u \in W_0$ , then  $u$  does not have a 2-neighbor and a weak neighbor, and so  $u$  might give charge  $\frac{1}{7}$  to each of neighbors by **R3** and **R4**. Thus the new charge of  $u$  satisfies

$$d^*(u) \geq 3 - 3 \times \frac{1}{7} = \frac{18}{7}.$$

Next, suppose that  $u \in W_1 \cup W_2 \cup W_3$ . Then  $u$  gives  $\frac{2}{7}$  to each of its 2-neighbors by **R1**.

(2-1). Suppose that  $u \in W_3$ .

By Lemma 3.2 ([C4]),  $u$  has three distinct weak neighbors  $x_1, x_2, x_3$  in  $W_1$ . Then  $u$  receives  $\frac{1}{7}$  from each  $x_i$  by **R2**. Since each  $x_i$  is not in  $W_2 \cup W_3$  and so  $u$  does not give any charge to them by **R2** or **R3**. Thus the new charge of  $u$  satisfies

$$d^*(u) \geq 3 - 3 \times \frac{2}{7} + 3 \times \frac{1}{7} = 3 - \frac{3}{7} = \frac{18}{7}.$$

(2-2). Suppose that  $u \in W_2$ .

Let  $x$  be the  $3^+$ -neighbor of  $u$ , and  $x_1$  and  $x_2$  be the two weak neighbors of  $u$ . See Figure 11 for an illustration. Then  $u$  receives  $\frac{1}{7}$  from  $x$  by **R3**. By [C4], each  $x_i$  is not in  $W_3$ . Thus  $u$  does not give any charge to them by **R2**. By [C3],  $x \notin W_2$  and so  $u$  does not send any charge

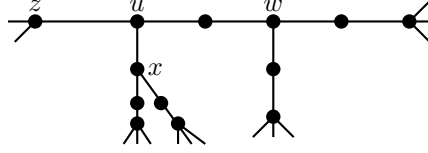


Figure 12: An illustration for case (2-3-2),  $u \in W_1$ ,  $x \in W_2$ , and  $w \in W_3$ .

to  $x$  by **R3**. Since  $u \notin W_0$ ,  $u$  does not send any charge to  $x$  by **R4**. Thus, in total,

$$d^*(u) \geq 3 - 2 \times \frac{2}{7} + \frac{1}{7} = 3 - \frac{3}{7} = \frac{18}{7}.$$

(2-3). Suppose that  $u \in W_1$ .

Let  $x$  and  $z$  be the two  $3^+$ -neighbors and  $w$  be the weak neighbor of  $u$ . Then by **[C5]**, we may assume that  $z \notin W_2$ . Thus  $u$  gives at most  $\frac{1}{7}$  to  $x$  and  $z$  in total by **R3**. By **R2**,  $u$  gives at most  $\frac{1}{7}$  to  $w$ .

(2-3-1). Suppose that  $x \notin W_2$  or  $w \notin W_3$ . If  $x \notin W_2$ , then  $u$  does not give any charge to  $x$  by **R3**. If  $w \notin W_3$ , then  $u$  does not give any charge to  $w$  by **R2**. Thus  $u$  gives at most  $\frac{1}{7}$  to  $x$ ,  $z$ , and  $w$  in total,

$$d^*(u) \geq 3 - \frac{2}{7} - \frac{1}{7} = \frac{18}{7}.$$

(2-3-2). Suppose that  $x \in W_2$  and  $w \in W_3$ . See Figure 12 for an illustration. Then  $u$  gives  $\frac{1}{7}$  to  $w$  by **R2**. By **[C6]**,  $z \notin W_1$ , which implies that  $z \in W_0$ . Then  $u$  receives  $\frac{1}{7}$  from  $z$  and  $u$  does not send any charge to  $z$  by **R4**. (Note that  $u$  is a vertex in  $W_1$ , which has a neighbor in  $W_2$  and one weak neighbor in  $W_3$ .) Thus

$$d^*(u) \geq 3 - \frac{2}{7} - 2 \cdot \frac{1}{7} + \frac{1}{7} = \frac{18}{7}.$$

(3) Suppose that  $\deg(u) \geq 4$ .

In this case,  $u$  gives charge at most  $\frac{2}{7}$  to its neighbors by **R1**, **R2** and **R3**. Note that any weak neighbor of  $u$  is not in  $W_3$  by **[C4]** and so  $u$  does not give any charge to its weak neighbor by **R2**. Thus

$$d^*(u) \geq \deg(u) - \deg(u) \times \frac{2}{7} = \frac{5}{7} \deg(u) > \frac{18}{7}.$$

This completes the proof of Theorem 1.3.

## 4 Proof of Theorem 1.4

We use the induction on the number of vertices and the number of edges. In the following, we let  $G$  be a minimal counterexample to Theorem 1.4. That is,  $G$  is a graph with the smallest number of vertices and edges,  $\text{mad}(G) < 14/5$ , and  $\text{ch}_3^d(G) \geq 8$ . Then there exists a list assignment  $L$  such that  $|L(v)| \geq 7$  for each  $v \in V(G)$  and  $G$  is not 3-dynamic  $L$ -colorable.

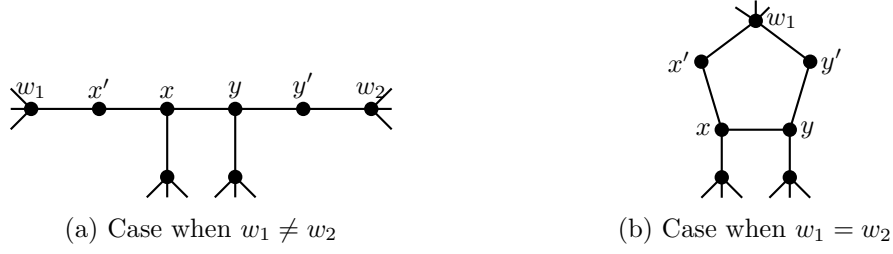


Figure 13: An illustration for Lemma 4.2.

**Lemma 4.1.** *For  $k \in \{3, 4\}$ , any  $k$ -vertex has at most  $(k - 2)$  2-neighbors.*

*Proof.* Let  $k \in \{3, 4\}$  and let  $v$  be a  $k$ -vertex, and  $v_1, v_2, \dots, v_k$  be its neighbors. Suppose that  $v$  has at least  $(k - 1)$  2-neighbors  $v_1, \dots, v_{k-1}$ . Let  $H = G - vv_k$ . Then  $\text{mad}(H) < 14/5$ . Since  $G$  is a minimal counterexample and  $H$  is smaller than  $G$ ,  $H$  is 3-dynamic  $L$ -colorable. Thus there is a 3-dynamic coloring  $\phi$  of  $H$  such that  $\phi(a) \in L(a)$  for any  $a \in V(H)$ . Then uncolor the vertex  $v$  and its 2-neighbors  $v_1, \dots, v_{k-1}$ .

Note that the number of forbidden colors at  $v$  is at most  $(k - 1) + 3 = k + 2 \leq 6$ . Thus  $v$  has at least one available color. We color  $v$  first with an available color. Then we recolor each 2-neighbor of  $v$  one by one. Since the number of available colors at each 2-neighbor of  $v$  is two, and so they are colorable so that  $v$  has three distinct colored neighbors. Thus  $G$  is 3-dynamic  $L$ -colorable, a contradiction.  $\square$

**Lemma 4.2.** *No two 3-vertices  $x$  and  $y$  in  $W_1$  are adjacent.*

*Proof.* Let  $x$  and  $y$  be 3-vertices such that  $x, y \in W_1$  and  $xy \in E(G)$ . Let  $x'$  and  $y'$  be 2-neighbors of  $x$  and  $y$ , respectively. And let  $w_1$  and  $w_2$  be the other neighbor of  $x'$  and  $y'$ , respectively. See Figure 13. Let  $H = G - \{x', y'\}$ . Then  $\text{mad}(H) < 14/5$ . Since  $G$  is a minimal counterexample and  $H$  is smaller than  $G$ ,  $H$  is 3-dynamic  $L$ -colorable. Thus there is a 3-dynamic coloring  $\phi$  of  $H$  such that  $\phi(a) \in L(a)$  for any  $a \in V(H)$ . Then uncolor the colors of  $x$  and  $y$ . Then the number of available colors at  $x$  is at least 3, and that of  $y$  is also at least 3. Color  $x$  with a color which is different from the color assigned at  $w_1$ , and  $y$  with a color which is different from the color assigned at  $w_2$ . Let  $L'(x')$  and  $L'(y')$  be the set of available colors at  $x'$  and  $y'$ , respectively.

Now, we consider two cases.

Case 1:  $w_1 \neq w_2$  (See (a) of Figure 13).

Since  $|L'(x')| \geq 1$  and  $|L'(y')| \geq 1$ , we can color  $x'$  and  $y'$  to have a dynamic 3-coloring.

Case 2:  $w_1 = w_2$  (See (b) of Figure 13).

If the degree of  $w_1$  in  $H$  is at least three, then  $x'$  and  $y'$  do not have to use different color and so we have a 3-dynamic coloring. Next, if the degree of  $w_1$  is 2 in  $H$ , then  $|L'(x')| \geq 2$  and  $|L'(y')| \geq 2$ . So they are colorable. Thus  $G$  is 3-dynamic  $L$ -colorable, a contradiction.  $\square$

**Lemma 4.3.** *No 3-vertex has three neighbors in  $W_1$ .*

*Proof.* Suppose that there is a vertex  $x$  having three neighbors  $x_1, x_2, x_3$  in  $W_1$ . Let  $x'_i$  be the 2-neighbor of  $x_i$  for each  $i \in [3]$ . See Figure 14 for an illustration. Let  $H = G -$

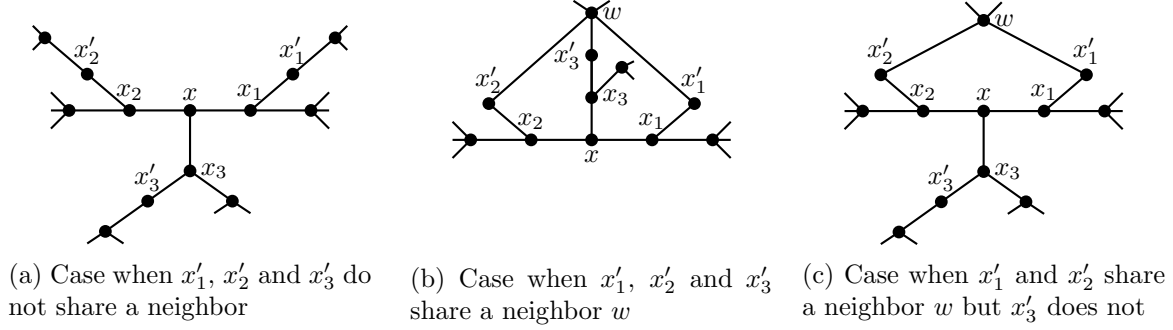


Figure 14: An illustration for Lemma 4.3.

$\{x, x_1, x_2, x_3, x'_1, x'_2, x'_3\}$ . Since  $G$  is a minimal counterexample and  $H$  is smaller than  $G$ ,  $H$  is 3-dynamic  $L$ -colorable. Thus there is a 3-dynamic coloring  $\phi$  of  $H$  such that  $\phi(a) \in L(a)$  for any  $a \in V(H)$ . Then the number of available colors at  $x$  is at least 4, that of  $x_i$  is at least 3 for each  $i \in [3]$ . We give a color to  $x_1, x_2, x_3, x$  with their available colors so that they get distinct colors. Then in the resulting coloring, the number of available colors at  $x'_i$  is at least 1. We color  $x'_1, x'_2, x'_3$  by that available colors. Here, the only thing that we have to concern is the case where  $x'_i$  and  $x'_j$  share a common neighbor and they get the same color.

Suppose that  $x'_1, x'_2, x'_3$  share a neighbor  $w$ . See (b) of Figure 14. Then  $w$  has at least three 2-neighbors and so by Lemma 4.1,  $w$  is a  $5^+$ -vertex. Thus in the 3-dynamic coloring  $\phi$  of  $H$ ,  $w$  has already at least two distinct colors in its neighbors other than the colors of  $x'_1, x'_2, x'_3$ . Thus eventually, the extended coloring of  $G$  results that  $G$  is 3-dynamic  $L$ -colorable, a contradiction.

Suppose that  $x'_1$  and  $x'_2$  share a neighbor  $w$  but  $x'_3$  does not. See (c) of Figure 14. Then  $w$  has at least two 2-neighbors by Lemma 4.1,  $w$  is a  $4^+$ -vertex. Thus in the 3-dynamic coloring  $\phi$  of  $H$ ,  $w$  has already at least two distinct colors in its neighbors other than the colors of  $x'_1$  and  $x'_2$ . Thus the extended coloring of  $G$  results that  $G$  is 3-dynamic  $L$ -colorable, a contradiction.  $\square$

We use discharging technique. We define the charge of each vertex  $v$  of  $G$  by its degree  $\deg(v)$ . Note that the average charge is less than  $\frac{14}{5}$ . Next, we distribute their charges by the following rules, and then we show that the new charge of each vertex is at least  $\frac{14}{5}$ , which leads a contradiction. The rules are as follows.

**R1.** A  $3^+$ -vertex gives  $\frac{2}{5}$  to its each of 2-neighbors.

**R2.** A  $3^+$ -vertex gives  $\frac{1}{10}$  to its each of 3-neighbors in  $W_1$ .

Let  $d^*(u)$  be the new charge after discharging. We will show that  $d^*(u) \geq \frac{14}{5}$  for all  $u \in V(G)$ . Note that by (1) of Lemma 2.1, each vertex of  $G$  is a  $2^+$ -vertex. If  $\deg(u) = 2$ , by (2) of Lemma 2.1, the neighbors of  $u$  are  $3^+$ -vertices and so it receives  $\frac{2}{5}$  from each of its neighbors by **R1**, which implies that

$$d^*(u) = 2 + \frac{2}{5} + \frac{2}{5} = \frac{14}{5}.$$

If  $\deg(u) = 3$ , then either  $u \in W_0$  or  $u \in W_1$  by Lemma 4.1. If  $u \in W_0$ ,  $u$  has at most two neighbors in  $W_1$  by Lemma 4.3 and so  $u$  gives  $\frac{1}{10}$  to each of its 3-neighbors in  $W_1$  by **R2** and so

$$d^*(u) \geq 3 - 2 \times \frac{1}{10} = \frac{14}{5}.$$

If  $u \in W_1$ , then by Lemma 4.2, the  $u$  has two  $3^+$ -neighbors and so it receives  $\frac{1}{10}$  from each of them by **R2** and so

$$d^*(u) \geq 3 - \frac{2}{5} + \frac{1}{10} + \frac{1}{10} = \frac{14}{5}.$$

If  $\deg(u) = 4$ , then by Lemma 4.1,  $u$  has at most two 2-neighbors, and so

$$d^*(u) \geq 4 - 2 \times \frac{2}{5} - 2 \times \frac{1}{10} = 3 > \frac{14}{5}.$$

If  $\deg(u) \geq 5$ , then

$$d^*(u) \geq \frac{3}{5} \deg(u) > \frac{14}{5}.$$

## 5 Proof of Theorem 1.5

We use the induction on the number of vertices and the number of edges. In the following, we let  $G$  be a minimal counterexample to Theorem 1.5. That is,  $G$  is a graph with the smallest number of vertices and edges,  $\text{mad}(G) < 3$ , and  $\text{ch}_3^d(G) \geq 9$ . Then there exists a list assignment  $L$  such that  $|L(v)| \geq 8$  for each  $v \in V(G)$  and  $G$  is not 3-dynamic  $L$ -colorable.

**Lemma 5.1.** *Any  $3^-$ -vertex has no 2-neighbors.*

*Proof.* Let  $x$  be a  $3^-$ -vertex and has a 2-neighbor  $y$ . Consider  $H = G - xy$ , deleting the edge  $xy$  from  $G$ . Then  $\text{mad}(H) < 3$ . Since  $G$  is a minimal counterexample and  $H$  is smaller than  $G$ ,  $H$  is 3-dynamic  $L$ -colorable. Thus there is a 3-dynamic coloring  $\phi$  of  $H$  such that  $\phi(a) \in L(a)$  for any  $a \in V(H)$ . Then uncolor the vertices  $x$  and  $y$ .

Note that the number of forbidden colors at  $x$  is at most  $3 + 3 + 1 = 7$ . Thus  $x$  has at least one available color. We color  $x$  first with that color. Then we recolor  $y$ , since the number of forbidden colors at  $y$  is at most  $3 + 3 = 6$ .  $\square$

**Lemma 5.2.** *For  $k \in \{4, 5\}$ , any  $k$ -vertex has at most  $(k - 2)$  2-neighbors.*

*Proof.* Let  $k \in \{4, 5\}$  and let  $v$  be a  $k$ -vertex, and  $v_1, v_2, \dots, v_k$  be its neighbors. Suppose that  $v$  has at least  $(k - 1)$  2-neighbors,  $v_1, \dots, v_{k-1}$ . Let  $H = G - vv_1$ . Then  $\text{mad}(H) < 3$ . Since  $G$  is a minimal counterexample and  $H$  is smaller than  $G$ ,  $H$  is 3-dynamic  $L$ -colorable. Thus there is a 3-dynamic 8-coloring  $\phi$  of  $H$  such that  $\phi(a) \in L(a)$  for any  $a \in V(H)$ . Then uncolor the vertex  $v$  and its all 2-neighbors.

Note that the number of forbidden colors at  $v$  is at most  $(k - 1) + 3 = k + 2 \leq 7$ . Thus  $v$  has at least one available color and we color  $v$  first with that color. Then we recolor each 2-neighbor of  $v$  one by one. Since the number of available colors at each 2-neighbor of  $v$  is 3, and so they are colorable so that  $v$  has three distinct colored neighbors.  $\square$



We use discharging technique. We define the charge of each vertex  $v$  of  $G$  by its degree  $\deg(v)$ . Note that the average charge is less than 3. Next, we distribute their charges by the following rules, and then we show that the new charge of each vertex is at least 3, which leads a contradiction. The rule is as follows.

**R1.** A  $4^+$ -vertex gives  $\frac{1}{2}$  to its each of 2-neighbors.

Let  $d^*(u)$  be the new charge after discharging. We will show that  $d^*(u) \geq 3$  for all  $u \in V(G)$ . By (1) of Lemma 2.1, each vertex of  $G$  is a  $2^+$ -vertex. If  $\deg(u) = 2$ , by Lemma 5.1, then the neighbors of  $u$  are  $4^+$ -vertices and so  $u$  receives  $\frac{1}{2}$  from each of its neighbors by **R1** and so

$$d^*(u) = 2 + \frac{1}{2} + \frac{1}{2} = 3.$$

If  $\deg(u) = 3$  then the charge of  $u$  is not changed and so  $d^*(u) = \deg(u) = 3$ . If  $\deg(u) = 4$  then by Lemma 5.2, it has at most two 2-neighbors and so

$$d^*(u) \geq 4 - 2 \times \frac{1}{2} = 3.$$

If  $\deg(u) = 5$  then by Lemma 5.2, it has at most three 2-neighbors and so

$$d^*(u) \geq 5 - 3 \times \frac{1}{2} > 3.$$

If  $\deg(u) \geq 6$ , then

$$d^*(u) \geq \deg(u) - \deg(u) \times \frac{1}{2} \geq \frac{\deg(u)}{2} \geq 3.$$

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